Boost-rotation symmetric vacuum spacetimes with spinning sources

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Boost-rotation symmetric vacuum spacetimes with spinning sources which correspond to gravitational field of uniformly accelerated spinning "particles" are studied. Regularity conditions and asymptotic properties are analyzed. News functions are derived by transforming the general spinning boost-rotation symmetric vacuum metric to Bondi-Sachs coordinates.

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I. INTRODUCTION AND SUMMARY

Boost-rotation symmetric spacetimes correspond to gravitational field of uniformly accelerated "particles". Usually conical singularities, which provide the "source" of the acceleration, are also present on the axis of the axial symmetry.

Boost-rotation symmetric spacetimes have two Killing vectors (the axial ξ and the boost η Killing vectors) and it has been proven that they are the only axially symmetric spacetimes with an additional symmetry that are radiative and admit global null infinity [1]. This result was generalized for spinning sources, i.e. for non-hypersurface orthogonal Killing vectors in [2]. Moreover boost-rotation symmetric spacetimes are the only radiative asymptotically flat spacetimes known in an analytical form which represent moving sources. While there are several known boost-rotation symmetric solutions with non-rotating sources (see [3, 4] and references therein), there is only one known exact solution with spinning sources – the spinning C-metric [5, 6, 7] corresponding to two uniformly accelerated Kerr black holes.

Thanks to the rotation of sources there appear torsion singularities besides conical singularities on the axis of the axial symmetry and consequently there can be regions with closed timelike curves (see [8] and also [6, 9] for examples and [9] for references).

The structure of a boost-rotation symmetric spacetime with hypersurface orthogonal Killing vectors was studied in [10] and the general form of its news function was found in [11, 12]. Recently, news functions for spinning boost-rotation symmetric Petrov type D spacetimes were computed in late time approximation in [13].

The present paper, where some results by Bičák and Bičák & Schmidt from [10, 11, 12] are generalized, is organized as follows. In Sec. II spinning boost-rotation symmetric (brs) vacuum spacetimes in coordinates adapted to the boost and rotation symmetries and null coordinates are examined, e.g. regularity of the spacetime on the roof and on the axis and asymptotic flatness at null infinity are studied. In Sec. III the spinning brs metric is transformed from the coordinates adapted to the boost and rotation symmetries to the Bondi-Sachs coordinates [14, 15, 16], suitable for examining radiation, to find the news functions of spinning brs spacetimes.

II. SPINNING BOOST-ROTATION SYMMETRIC SPACETIMES – REGULARITY CONDITIONS, ASYMPTOTIC BEHAVIOUR

The general form of spinning brs metric in coordinates adapted to the boost and rotation symmetries $\{t,\ \rho,\ z,\ \varphi\}$ is [6]

$$\begin{aligned}\mathrm{d}s^2 \;=\; & -\mathrm{e}^{\lambda}\mathrm{d}\rho^2 - \rho^2\mathrm{e}^{-\mu}\mathrm{d}\varphi^2 + 2a\mathrm{e}^{\mu}(z\mathrm{d}t - t\mathrm{d}z)\mathrm{d}\varphi + a^2\mathrm{e}^{\mu}(z^2-t^2)\mathrm{d}\varphi^2 \\ & -\frac{1}{z^2-t^2}[(\mathrm{e}^{\lambda}z^2 - \mathrm{e}^{\mu}t^2)\mathrm{d}z^2 - 2zt(\mathrm{e}^{\lambda}-\mathrm{e}^{\mu})\mathrm{d}z\ \mathrm{d}t - (\mathrm{e}^{\mu}z^2 - \mathrm{e}^{\lambda}t^2)\mathrm{d}t^2]\ , \end{aligned} \qquad (1)$$

where μ, λ, and a are functions of

$$A = \rho^2, \quad B = z^2 - t^2\ .$$

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It has two Killing vectors

$$\xi = \frac{\partial}{\partial\varphi}\ , \quad \eta = t\frac{\partial}{\partial z} + z\frac{\partial}{\partial t} \tag{2}$$

with norms

$$\begin{aligned}\xi^\alpha\xi_\alpha &= g_{\varphi\varphi} = -\rho^2 \mathrm{e}^{-\mu} + a^2(z^2-t^2)\mathrm{e}^{\mu} = -A\mathrm{e}^{-\mu} + a^2 B\mathrm{e}^{\mu}\ ,\\ \eta^\alpha\eta_\alpha &= g_{tt}z^2 + g_{zz}t^2 + 2g_{zt}zt = B\mathrm{e}^{\mu}\ .\end{aligned}$$

As in the non-spinning case [10], two null hyperplanes $B=0$, i.e. $z=\pm t$, will be called the "roof", the points with $A=0$ the "axis", the region of the spacetime with $B<0$ "above the roof", and finally the region with $B>0$ "bellow the roof". Notice that the behaviour of the boost and axial Killing vectors (2) is more complicated in the spinning case. Bellow the roof ($B>0$), the boost Killing vector η is mostly timelike as in the non-spinning case but in the vicinity of spinning sources there may also occur ergoregions where it is spacelike. Due to the presence of spinning strings there may be also regions in their neighbourhood with closed timelike curves where the axial Killing vector ξ is timelike. In order to determine if there exist both timelike and spacelike Killing vectors everywhere bellow the roof ($B>0$), we study a general linear combination of the boost and the axial Killing vectors with constant coefficients $X = c_1\xi + c_2\eta$. Its norm may be both positive and negative if the product of eigenvalues of the quadratic form (${c_1}^2 g_{\phi\phi} + \ldots$) given by the norm is negative, i.e. if $-\rho^2 B<0$, which is satisfied everywhere bellow the roof, where the spacetime is thus stationary and may be transformed to the stationary Weyl metric (A1) (see e.g. [6]). However, above the roof ($B<0$), the product is everywhere positive $-\rho^2 B>0$ and thus there does not exist a timelike Killing vector and the spacetime is nonstationary there.[1]

Vacuum Einstein's equations for the spinning brs metric (1) are

$$\begin{aligned}A\mu_{,AA} + B\mu_{,BB} + \mu_{,A} + \mu_{,B} &= -\frac{B}{A}\,\mathrm{e}^{2\mu}(A{a_{,A}}^2 + B{a_{,B}}^2)\ , &(3)\\ 0 &= AB\Big(\mathrm{e}^{2\mu}a_{,A}\Big)_{,A} + \Big(B^2\mathrm{e}^{2\mu}a_{,B}\Big)_{,B}\ , &(4)\\ (A+B)\lambda_{,A} &= (A-B)\mu_{,A} - 2B\mu_{,B} - B(B{\mu_{,B}}^2 - A{\mu_{,A}}^2) + 2AB\mu_{,A}\,\mu_{,B}\\ &\quad + \frac{B^2}{A}\mathrm{e}^{2\mu}(B{a_{,B}}^2 - A{a_{,A}}^2 - 2Aa_{,A}\,a_{,B})\ , &(5)\\ (A+B)\lambda_{,B} &= (A-B)\mu_{,B} + 2A\mu_{,A} + A(B{\mu_{,B}}^2 - A{\mu_{,A}}^2) + 2AB\mu_{,A}\,\mu_{,B}\\ &\quad - B\mathrm{e}^{2\mu}(B{a_{,B}}^2 - A{a_{,A}}^2 + 2Ba_{,A}\,a_{,B})\ . &(6)\end{aligned}$$

Notice that Eqs. (3), (4) are integrability conditions for Eqs. (5), (6).

First let us investigate the regularity of the roof and the axis. From Eq. (5) it follows that on the roof (i.e. for $B=0$)

$$\lambda_{,A}(A,0) = \mu_{,A}(A,0) \quad \rightarrow \quad \lambda(A,0) - \mu(A,0) = K_1 = \text{const}\ .$$

The roof is regular (i.e. g_{zz}, g_{tt}, and g_{tz} in (1) are nonsingular on the roof) if for $B=0$

$$\lambda(A,0) = \mu(A,0)\ , \quad \text{i.e.} \quad K_1 = 0\ . \tag{7}$$

From Eqs. (3), (5), and (6) on the axis ($A=0$) we get

$$\begin{aligned}a_{,B}(0,B) &= 0 \quad \rightarrow \quad a = \tilde{a}_0 + \tilde{a}_1(B)A + \mathcal{O}(A^2)\ , \quad \tilde{a}_0 = \text{const}\ ,\\ \lambda_{,B}(0,B) + \mu_{,B}(0,B) &= 0 \quad \rightarrow \quad \lambda(0,B) + \mu(0,B) = K_2 = \text{const}\ .\end{aligned}$$

The axis regularity condition

$$\lim_{\rho_0\to 0}\frac{1}{2\pi}\frac{\int_0^{2\pi}\sqrt{g_{\varphi\varphi}|_{\rho_0}}\mathrm{d}\varphi}{\int_0^{\rho_0}\sqrt{g_{\rho\rho}}\mathrm{d}\rho} = 1$$

---

[1] It is easier to perform these calculations in coordinates $\{\gamma,\ \rho,\ \beta,\ \varphi\}$ and $\{b,\ \rho,\ \chi,\ \varphi\}$, given in App. A, for regions bellow and above the roof, respectively.

(or equivalently g_{xx}, g_{yy} are nonsingular and $g_{xy} = 0$ there, see (A5) in App. A) is satisfied if

$$
\begin{aligned}
a(0,B) &= 0\ , \quad \rightarrow \quad a = \tilde{a}_1(B)A + \mathcal{O}(A^2)\ , \quad \text{i.e.} \quad \tilde{a}_0 = 0\ , & (8)\\
\lambda(0,B)+\mu(0,B) &= 0\ , \quad \text{i.e.} \quad K_2 = 0\ . & (9)
\end{aligned}
$$

If $K_2 \neq 0$ there is a conical singularity along the axis and if $\tilde{a}_0 \neq 0$ a torsion singularity is present there. The regularity condition of the roof (7) is the same as for nonspinning brs spacetimes [10], however, a new condition (8) arises for the regularity of the axis except for (9) which also appears in the nonspinning case [10].

Now let us turn our attention to asymptotic behaviour of spinning brs spacetimes at null infinity. For this purpose we transform (1) to null coordinates in two steps: first transforming it to coordinates $\{b,\ \rho,\ \chi,\ \varphi\}$ by (3.10) in [10]

$$b = \sqrt{-B} = \sqrt{t^2 - z^2}\ , \quad \tanh\chi = \pm\frac{z}{t}$$

we obtain the metric

$$\mathrm{d}s^2 = \mathrm{e}^{\lambda}(\mathrm{d}b^2 - \mathrm{d}\rho^2) - \rho^2\mathrm{e}^{-\mu}\mathrm{d}\varphi^2 - b^2\mathrm{e}^{\mu}(\mathrm{d}\chi + a\mathrm{d}\varphi)^2\ . \tag{10}$$

Finally transforming (10) to coordinates $\{\bar{u},\ \bar{v},\ \chi,\ \varphi\}$ by (3.15) in [10]

$$\bar{u} = b - \rho\ , \quad \bar{v} = b + \rho$$

we obtain

$$\mathrm{d}s^2 = \mathrm{e}^{\lambda}\mathrm{d}\bar{u}\mathrm{d}\bar{v} - \frac{(\bar{v}-\bar{u})^2}{4}\mathrm{e}^{-\mu}\mathrm{d}\varphi^2 - \frac{(\bar{v}+\bar{u})^2}{4}\mathrm{e}^{\mu}(\mathrm{d}\chi + a\mathrm{d}\varphi)^2\ . \tag{11}$$

Vacuum Einstein's equations for (11) read

$$
\begin{aligned}
\mu_{,\bar{u}\bar{v}} + \frac{1}{\bar{v}^2-\bar{u}^2}(\bar{v}\mu_{,\bar{u}} - \bar{u}\mu_{,\bar{v}}) &= \left(\frac{\bar{v}+\bar{u}}{\bar{v}-\bar{u}}\right)^2 \mathrm{e}^{2\mu} a_{,\bar{u}}\, a_{,\bar{v}}\ ,\\
0 &= a_{,\bar{u}\bar{v}} + a_{,\bar{u}}\left(\mu_{,\bar{v}} + \frac{\bar{v}-2\bar{u}}{\bar{v}^2-\bar{u}^2}\right) + a_{,\bar{v}}\left(\mu_{,\bar{u}} + \frac{2\bar{v}-\bar{u}}{\bar{v}^2-\bar{u}^2}\right)\ ,\\
-\bar{u}\lambda_{,\bar{u}} &= \bar{v}\mu_{,\bar{u}} + \frac{\bar{v}^2-\bar{u}^2}{4}\mu_{,\bar{u}}{}^2 + \frac{(\bar{v}+\bar{u})^3}{\bar{v}-\bar{u}}\frac{\mathrm{e}^{2\mu}}{4}a_{,\bar{u}}{}^2\ ,\\
-\bar{v}\lambda_{,\bar{v}} &= \bar{u}\mu_{,\bar{v}} - \frac{\bar{v}^2-\bar{u}^2}{4}\mu_{,\bar{v}}{}^2 - \frac{(\bar{v}+\bar{u})^3}{\bar{v}-\bar{u}}\frac{\mathrm{e}^{2\mu}}{4}a_{,\bar{v}}{}^2\ .
\end{aligned}
\tag{12}
$$

Assuming the metric functions μ, λ, and a to have expansions in $\bar{v}^{-1}$ for $\bar{v} \to \infty$ ($\mu(\bar{u},\ \bar{v}) = \mu_0(\bar{u}) + \mu_1(\bar{u})/\bar{v} + \ldots$) and solving Eqs. (12) at null infinity, i.e. for the limit $\bar{v} \to \infty$ and $\bar{u}$, χ, φ constant, we get

$$
\begin{aligned}
\mu &= \mu_0 + \frac{\mu_1(\bar{u})}{\bar{v}} + \mathcal{O}(\bar{v}^{-2})\ ,\\
\lambda &= \lambda_0(\bar{u}) + \frac{\lambda_1(\bar{u})}{\bar{v}} + \mathcal{O}(\bar{v}^{-2})\ ,\\
a &= a_0 + \frac{a_1(\bar{u})}{\bar{v}} + \mathcal{O}(\bar{v}^{-2})\ ,
\end{aligned}
\tag{13}
$$

where a_0 and μ_0 are constants and $\lambda_0(\bar{u})$ satisfies

$$\lambda_{0,\bar{u}} = -\frac{1}{4\bar{u}}\left(4\mu_{1,\bar{u}} + \mu_{1,\bar{u}}{}^2 + \mathrm{e}^{2\mu_0}a_{1,\bar{u}}{}^2\right)\ .$$

The metric (11) with the metric functions (13) is asymptotically Minkowskian at null infinity as in the limit $\bar{v} \to \infty$ and $\bar{u}$, χ, φ constant, it can be transformed to the Minkowski metric using the transformations (3.23), (3.24) in [10]

$$\bar{u}' = \mathrm{e}^{\frac{1}{2}\mu_0}\int \mathrm{e}^{\lambda_0(\bar{u})}\mathrm{d}\bar{u}\ , \quad \bar{v}' = \mathrm{e}^{-\frac{1}{2}\mu_0}\bar{v}\ , \quad \chi' = \mathrm{e}^{\mu_0}\chi$$

and

$$\chi'' = \chi' + a_0\mathrm{e}^{\mu_0}\varphi\ .$$

III. THE BONDI-SACHS COORDINATES AND NEWS FUNCTIONS FOR SPINNING BRS SPACETIMES

In this section we transform the spinning brs metric (1) into the Bondi-Sachs coordinates $\{u,\ r,\ \theta,\ \phi\}$, in which the metric, that does not depend on ϕ because of the axial symmetry, has the form [14, 15, 16]

$$\mathrm{d}s^2 = g_{uu}\mathrm{d}u^2 + 2g_{ur}\mathrm{d}u\mathrm{d}r + 2g_{u\theta}\mathrm{d}u\mathrm{d}\theta + 2g_{u\phi}\mathrm{d}u\mathrm{d}\phi + g_{\theta\theta}\mathrm{d}\theta^2 + g_{\phi\phi}\mathrm{d}\phi^2 + 2g_{\theta\phi}\mathrm{d}\theta\mathrm{d}\phi \tag{14}$$

with the following expansion for $r \to \infty$ and u, θ, and ϕ constant

$$\begin{aligned}
g_{uu} &= 1 - \frac{2M}{r} + \mathcal{O}(r^{-2}) \ ,\\
g_{ur} &= 1 - \frac{c^2+d^2}{2r^2} + \mathcal{O}(r^{-4}) \ ,\\
g_{u\theta} &= -(c_{,\theta} + 2c\cot\theta) + \mathcal{O}(r^{-1}) \ ,\\
g_{u\phi} &= -(d_{,\theta} + 2d\cot\theta)\sin\theta + \mathcal{O}(r^{-1}) \ ,\\
g_{\theta\theta} &= -r^2 - 2cr - 2(c^2+d^2) + \mathcal{O}(r^{-1}) \ ,\\
g_{\theta\phi} &= -2dr\sin\theta + \mathcal{O}(r^0) \ ,\\
g_{\phi\phi} &= -r^2\sin^2\theta + 2cr\sin^2\theta - 2(c^2+d^2)\sin^2\theta + \mathcal{O}(r^{-1}) \ ,
\end{aligned} \tag{15}$$

where c, d, M are functions of u and θ. As a consequence of Einstein's equations, time dependence of the mass aspect M is determined by the news functions $c_{,u}$ and $d_{,u}$ [15, 16]

$$M_{,u} = -({c_{,u}}^2 + {d_{,u}}^2) + \tfrac{1}{2}(c_{,\theta\theta} + 3c_{,\theta}\cot\theta - 2c)_{,u} \ .$$

If there is nonvanishing news function, gravitational radiation is present and the total Bondi mass at future null infinity is decreasing.

In order to find the transformation of spinning brs spacetimes from the coordinates $\{t,\ \rho,\ z,\ \varphi\}$ with the metric (1) into the Bondi-Sachs coordinates $\{u,\ r,\ \theta,\ \phi\}$ with the metric (14) and its expansions (15) we follow [11] and we first transform the metric (1) to flat-space spherical coordinates $\{R,\ \Theta,\ \varphi\}$ and a flat-space retarded time U using the relations

$$\begin{aligned}
t &= U + R \ ,\\
\rho &= R\sin\Theta \ ,\\
z &= R\cos\Theta \ .
\end{aligned} \tag{16}$$

We assume the metric functions to have the expansions in powers R^{-1}

$$\begin{aligned}
\lambda(U,\ \Theta) &= \lambda_0(U,\ \Theta) + \frac{\lambda_1(U,\ \Theta)}{R} + \mathcal{O}(R^{-2}) \ ,\\
\mu(U,\ \Theta) &= \mu_0 + \frac{\mu_1(U,\ \Theta)}{R} + \mathcal{O}(R^{-2}) \ ,\\
a(U,\ \Theta) &= a_0 + \frac{a_1(U,\ \Theta)}{R} + \mathcal{O}(R^{-2}) \ ,
\end{aligned} \tag{17}$$

where μ_0 and a_0 are constants and thus

$$\begin{aligned}
\mathrm{e}^{\lambda} &= \beta(U,\ \Theta)\Big(1 + \frac{\lambda_1(U,\ \Theta)}{R} + \mathcal{O}(R^{-2})\Big) \ ,\\
\mathrm{e}^{\mu} &= \alpha\Big(1 + \frac{\mu_1(U,\ \Theta)}{R} + \mathcal{O}(R^{-2})\Big)
\end{aligned} \tag{18}$$

with

$$\begin{aligned}
\beta(U,\ \Theta) &= \mathrm{e}^{\lambda_0(U,\ \Theta)} \ ,\\
\alpha &= \mathrm{e}^{\mu_0} \ .
\end{aligned} \tag{19}$$

Now we transform the metric further to the Bondi-Sachs coordinates by an asymptotic transformation

$$
\begin{aligned}
U &= \overset{o}{\pi}(u,\theta) + \frac{\overset{1}{\pi}(u,\theta)}{r} + \frac{\overset{2}{\pi}(u,\theta)}{r^2} + \mathcal{O}(r^{-3}) \ , \\
R &= q(u,\theta)r + \overset{o}{\sigma}(u,\theta) + \frac{\overset{1}{\sigma}(u,\theta)}{r} + \mathcal{O}(r^{-2}) \ , \\
\Theta &= \overset{o}{\tau}(u,\theta) + \frac{\overset{1}{\tau}(u,\theta)}{r} + \frac{\overset{2}{\tau}(u,\theta)}{r^2} + \mathcal{O}(r^{-3}) \ , \\
\varphi &= \phi + \overset{o}{f}(u,\theta) + \frac{\overset{1}{f}(u,\theta)}{r} + \frac{\overset{2}{f}(u,\theta)}{r^2} + \mathcal{O}(r^{-3}) \ .
\end{aligned}
\tag{20}
$$

Comparing the resulting metric expansions with the expansions (15) we obtain differential equations for coefficients entering the asymptotic transformation (20). Since these equations are lenghty we present only their solutions in App. B. Their integrability condition (obtained comparing (B4) and (B5)) turns out to be the same as in the non-spinning case [11]

$$
\beta_{,\overset{o}{\pi}}\,\overset{o}{\pi} + \beta_{,\overset{o}{\tau}}\tan\overset{o}{\tau} = 0 \ , \quad \text{or equivalently} \quad \lambda_{0,U}\,U + \lambda_{0,\Theta}\tan\Theta = 0 \ ,
$$

where we used the relations $\beta_{,u} = \beta_{,\overset{o}{\pi}}\,\overset{o}{\pi}_{,u}$ and $\beta_{,\theta} = \beta_{,\overset{o}{\tau}}\,\overset{o}{\tau}_{,\theta} + \beta_{,\overset{o}{\pi}}\,\overset{o}{\pi}_{,\theta}$. Solving Eqs. (B1), (B2), (B3), and (B6) one may infer the first order coefficients in the expansions (20)

$$
\overset{o}{\tau} = 2\arctan\Big[e^{-\nu}(\tan\tfrac{1}{2}\theta)^K\Big] \ , \tag{21}
$$

$$
q = \frac{1}{\sqrt{K}}\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta\Big[e^{\nu}(\cot\tfrac{1}{2}\theta)^K + e^{-\nu}(\tan\tfrac{1}{2}\theta)^K\Big] \ , \tag{22}
$$

$$
\overset{o}{f} = \frac{a_0\alpha}{K}\ln\left(\frac{\sin\overset{o}{\tau}}{1+\cos\overset{o}{\tau}}\right) \ , \tag{23}
$$

$$
\overset{o}{\pi}_{,u} = \frac{1}{\beta q} \ , \tag{24}
$$

where $K \equiv \frac{1+a_0^2\alpha^2}{\alpha}$ and ν is an arbitrary constant. The axis (which is the same in both coordinates, i.e. $\Theta = 0, \pi \to \theta = 0, \pi$) is singular for $K \neq 1$ as q goes to 0 for $K < 1$ and to ∞ for $K > 1$ there. Since the coordinate system $\{t, \rho, z, \varphi\}$ can be chosen in such a way that $a_0 = 0$, we present here news functions for $a_0 = 0$ and the case $a_0 \neq 0$ is given in App. B. From Eqs. (B7) and (B8) we obtain the news functions

$$
c_{,u} = \tfrac{1}{2}\mu_{1,u} - \frac{{q_{,\theta}}^2}{2q^2} - \frac{q_{,\theta}\cot\overset{o}{\tau}}{q^2\sqrt{\alpha}} + \frac{1}{2q^2\beta\sin^2\overset{o}{\tau}} - \frac{1}{2} - \frac{\cot^2\overset{o}{\tau}}{2q^2\alpha} \ , \tag{25}
$$

$$
d_{,u} = -\tfrac{1}{2}\alpha a_{1,u} \ . \tag{26}
$$

Having the news functions of the system one can compute the Bondi mass, see [2].

For a special case $K = \alpha = 1$, i.e. for a regular axis, we get from (21), (22), and (23)

$$
\begin{aligned}
\overset{o}{\tau} &= 2\arctan\Big(e^{-\nu}\tan\tfrac{1}{2}\theta\Big) \ , \\
q &= \cosh\nu + \cos\theta\sinh\nu \ , \\
\overset{o}{f} &= 0 \ .
\end{aligned}
\tag{27}
$$

Coordinate systems with different ν are connected by Lorentz transformations along the axis belonging to the Bondi-Metzner-Sachs group and thus as in [11] we may without loss of generality put $\nu = 1$ which implies $q = 1$ and $\overset{o}{\tau} = \theta$. Then the coefficient $\overset{o}{\pi}$ can be computed from the relation

$$
\int e^{\lambda_0(\overset{o}{\pi},\,\theta)}\,d\,\overset{o}{\pi} = u + \omega(\theta) \tag{28}
$$

obtained from Eq. (24). The function $\omega(\theta)$ in (28) represents a supertranslation also belonging to the Bondi-Metzner-Sachs group and thus it may be again put equal to zero without loss of generality. Finally the news functions (25) and (26) read

$$\begin{aligned} c_{,u} &= -\frac{1}{2\sin^2\theta} + \frac{1}{2\beta\sin^2\theta} + \tfrac{1}{2}\mu_{1,u} = \frac{1}{2\beta\sin^2\theta}(1-\beta+\mu_{1,u}\,\beta\sin^2\theta)\ , && (29)\\ d_{,u} &= -\tfrac{1}{2}a_{1,u}\ . && (30)\end{aligned}$$

For $a_1 = 0$ (29) and (30) reduce to news functions as given in [11, 12] for the nonrotating case.

APPENDIX A: COORDINATE SYSTEMS ADAPTED TO THE BOOST AND ROTATION SYMMETRIES

In the nonradiative stationary region bellow the roof, spinning brs metric can be transformed to the stationary Weyl coordinates $\{\bar t,\ \bar\rho,\ \bar z,\ \bar\varphi\}$ with the Killing vectors $\xi = \partial_{\bar\varphi}$, $\eta = \partial_{\bar t}$ and the metric

$$\mathrm{d}s^2 = -\mathrm{e}^{-2U}[\mathrm{e}^{2\nu}(\mathrm{d}\bar\rho^2+\mathrm{d}\bar z^2)+\bar\rho^2\mathrm{d}\bar\varphi^2] + \mathrm{e}^{2U}(\mathrm{d}\bar t + a\mathrm{d}\bar\varphi)^2\ . \qquad \text{(A1)}$$

Vacuum Einstein's equations have the form [17]

$$\begin{aligned} U_{,\bar\rho\bar\rho} + U_{,\bar z\bar z} + \frac{U_{,\bar\rho}}{\bar\rho} &= -\frac{\mathrm{e}^{4U}}{2\bar\rho^2}({a_{,\bar\rho}}^2 + {a_{,\bar z}}^2)\ ,\\ 0 &= \Big(\mathrm{e}^{4U}\frac{a_{,\bar\rho}}{\bar\rho}\Big)_{,\bar\rho} + \Big(\mathrm{e}^{4U}\frac{a_{,\bar z}}{\bar\rho}\Big)_{,\bar z}\ ,\\ \frac{\nu_{,\bar\rho}}{\bar\rho} &= {U_{,\bar\rho}}^2 - {U_{,\bar z}}^2 - \frac{\mathrm{e}^{4U}}{4\bar\rho^2}({a_{,\bar\rho}}^2 - {a_{,\bar z}}^2)\ , \qquad \text{(A2)}\\ \frac{\nu_{,\bar z}}{\bar\rho} &= 2U_{,\bar\rho}U_{,\bar z} - \frac{\mathrm{e}^{4U}}{2\bar\rho^2}a_{,\bar\rho}a_{,\bar z}\ .\end{aligned}$$

Another appropriate coordinate system in the stationary region bellow the roof is $\{\gamma,\ \rho,\ \beta,\ \varphi\}$ with Killing vectors $\xi = \partial_\varphi$, $\eta = \partial_\gamma$ and the metric

$$\mathrm{d}s^2 = -\mathrm{e}^{\lambda}(\mathrm{d}\rho^2+\mathrm{d}\beta^2) - \rho^2\mathrm{e}^{-\mu}\mathrm{d}\varphi^2 + \beta^2\mathrm{e}^{\mu}(\mathrm{d}\gamma + a\mathrm{d}\varphi)^2\ , \qquad \text{(A3)}$$

connected with the stationary Weyl coordinates by (see (5.4), (5.6) in [10])

$$\bar t = \gamma\ ,\quad \bar\rho = \rho\beta\ ,\quad \bar z - \bar z_0 = \frac{\beta^2-\rho^2}{2}\ ,\quad \bar\varphi = \varphi\ ,\quad \bar z_0 = \text{const}\ ,\quad \mathrm{e}^{2U} = \beta^2\mathrm{e}^{\mu}\ ,\quad \mathrm{e}^{2\nu} = \frac{\beta^2}{\rho^2+\beta^2}\mathrm{e}^{\mu+\lambda}\ .$$

Vacuum Einstein's equations read

$$\begin{aligned} \mu_{,\rho\rho} + \mu_{,\beta\beta} + \frac{\mu_{,\rho}}{\rho} + \frac{\mu_{,\beta}}{\beta} &= -\frac{\beta^2}{\rho^2}\mathrm{e}^{2\mu}({a_{,\rho}}^2 + {a_{,\beta}}^2)\ ,\\ 0 &= \Big(\frac{\beta^3}{\rho}\mathrm{e}^{2\mu}a_{,\rho}\Big)_{,\rho} + \Big(\frac{\beta^3}{\rho}\mathrm{e}^{2\mu}a_{,\beta}\Big)_{,\beta}\ ,\\ (\rho^2+\beta^2)\lambda_{,\rho} &= (\rho^2-\beta^2)\mu_{,\rho} - 2\rho\beta\mu_{,\beta} - \tfrac{1}{2}\rho\beta^2({\mu_{,\beta}}^2 - {\mu_{,\rho}}^2) + \rho^2\beta\mu_{,\rho}\,\mu_{,\beta} \qquad \text{(A4)}\\ &\quad + \frac{\beta^4}{2\rho}\mathrm{e}^{2\mu}\Big({a_{,\beta}}^2 - {a_{,\rho}}^2 - 2\frac{\rho}{\beta}a_{,\rho}\,a_{,\beta}\Big)\ ,\\ (\rho^2+\beta^2)\lambda_{,\beta} &= (\rho^2-\beta^2)\mu_{,\beta} + 2\rho\beta\mu_{,\rho} + \tfrac{1}{2}\rho^2\beta({\mu_{,\beta}}^2 - {\mu_{,\rho}}^2) + \rho\beta^2\mu_{,\rho}\,\mu_{,\beta}\\ &\quad - \tfrac{1}{2}\beta^3\mathrm{e}^{2\mu}\Big({a_{,\beta}}^2 - {a_{,\rho}}^2 + 2\frac{\beta}{\rho}a_{,\rho}\,a_{,\beta}\Big)\ .\end{aligned}$$

The stationary region of a brs spacetime, under the roof, is composed of two identical regions ($z > 0$, $z > |t|$ and $z < 0$, $z < -|t|$) and each of them can be transformed to coordinates $\{\bar t,\ \bar\rho,\ \bar z,\ \bar\varphi\}$ or $\{\gamma,\ \rho,\ \beta,\ \varphi\}$.

By further transformation (3.5) in [10] to coordinates $\{t,\ \rho,\ z,\ \varphi\}$

$$\tanh\gamma = \pm\frac{t}{z}\ ,\quad \beta = \sqrt{z^2-t^2}\ ,\quad B \equiv \beta^2\ ,\quad A \equiv \rho^2\ ,$$

we arrive at the metric (1) where nonstationary region above the roof (again composed of two identical regions) appears as in the nonspinning case [10].

For examining regularity of the axis it is convenient to transform (1) to coordinates $\{t,\, x,\, y,\, z\}$, where $x = \rho\cos\varphi$, $y = \rho\sin\varphi$:

$$
\begin{aligned}
\mathrm{d}s^2 \;=\; & -\frac{1}{x^2+y^2}\Big[(\mathrm{e}^{\lambda}x^2+\mathrm{e}^{-\mu}y^2)\mathrm{d}x^2+(\mathrm{e}^{\lambda}y^2+\mathrm{e}^{-\mu}x^2)\mathrm{d}y^2+2xy(\mathrm{e}^{\lambda}-\mathrm{e}^{-\mu})\mathrm{d}x\mathrm{d}y \\
& -\frac{z^2-t^2}{x^2+y^2}a^2\mathrm{e}^{\mu}(-y\mathrm{d}x+x\mathrm{d}y)^2-2a\mathrm{e}^{\mu}(-yz\mathrm{d}x\mathrm{d}t+yt\mathrm{d}x\mathrm{d}z+xz\mathrm{d}y\mathrm{d}t-xt\mathrm{d}y\mathrm{d}z)\Big] \\
& -\frac{1}{z^2-t^2}[(\mathrm{e}^{\lambda}z^2-\mathrm{e}^{\mu}t^2)\mathrm{d}z^2-2zt(\mathrm{e}^{\lambda}-\mathrm{e}^{\mu})\mathrm{d}z\mathrm{d}t-(\mathrm{e}^{\mu}z^2-\mathrm{e}^{\lambda}t^2)\mathrm{d}t^2]\ .
\end{aligned}
\tag{A5}
$$

Let us finally write down vacuum Einstein's equations for the metric (10) with the Killing vectors $\xi = \partial_\varphi$, $\eta = \partial_\chi$

$$
\begin{aligned}
\mu_{,\rho\rho}-\mu_{,bb}+\frac{\mu_{,\rho}}{\rho}-\frac{\mu_{,b}}{b} &= \frac{b^2}{\rho^2}\mathrm{e}^{2\mu}({a_{,\rho}}^2-{a_{,b}}^2)\ , \\
0 &= \Big(\frac{b^3}{\rho}\mathrm{e}^{2\mu}a_{,\rho}\Big)_{,\rho}-\Big(\frac{b^3}{\rho}\mathrm{e}^{2\mu}a_{,b}\Big)_{,b}\ , \\
(\rho^2-b^2)\lambda_{,\rho} &= (\rho^2+b^2)\mu_{,\rho}-2\rho b\mu_{,b}-\tfrac{1}{2}\rho b^2({\mu_{,b}}^2+{\mu_{,\rho}}^2)+\rho^2 b\mu_{,\rho}\,\mu_{,b} \\
&\quad+\frac{b^4}{2\rho}\mathrm{e}^{2\mu}\Big(-{a_{,b}}^2-{a_{,\rho}}^2+2\frac{\rho}{b}a_{,\rho}\,a_{,b}\Big)\ , \\
(\rho^2-b^2)\lambda_{,b} &= -(\rho^2+b^2)\mu_{,b}+2\rho b\mu_{,\rho}-\tfrac{1}{2}\rho^2 b({\mu_{,b}}^2+{\mu_{,\rho}}^2)+\rho b^2\mu_{,\rho}\,\mu_{,b} \\
&\quad+\frac{b^3}{2}\mathrm{e}^{2\mu}\Big(-{a_{,b}}^2-{a_{,\rho}}^2+2\frac{b}{\rho}a_{,\rho}\,a_{,b}\Big)\ .
\end{aligned}
\tag{A6}
$$

The coordinates $\{b,\, \rho,\, \chi,\, \varphi\}$ for the nonstationary region above the roof are analogical to coordinates $\{\gamma,\ \rho,\ \beta,\ \varphi\}$ (A3) in the stationary region bellow the roof.

As for (3)–(6), in each set of Einstein's equations (A2), (A4), and (A6), the first two are integrability conditions for the other two.

APPENDIX B: TRANSFORMATION OF THE SPINNING BRS METRIC TO THE BONDI-SACHS COORDINATES

The spinning brs metric (1) with expansions (17)–(19) being transformed to the Bondi-Sachs coordinates with the metric (14) using transformations (16), (20) and compared with (15) leads to lenghty equations for coefficients of the asymptotic transformation (20) and metric functions from (15). We present here only their solutions:

$$
\begin{aligned}
(g_{u\phi},\ r^2) &= 0 && \rightarrow \overset{o}{f}_{,u} = \overset{o}{\tau}_{,u}\frac{a_0\alpha}{K\sin\overset{o}{\tau}}\ , \\
(g_{uu},\ r^2) &= 0 && \rightarrow \overset{o}{\tau}_{,u} = 0 = \overset{o}{f}_{,u}\ , \\
(g_{uu},\ r^1) &= 0 && \rightarrow q_{,u} = 0\ , \\
(g_{\theta\phi},\ r^2) &= 0 && \rightarrow \overset{o}{f}_{,\theta} = \overset{o}{\tau}_{,\theta}\frac{a_0\alpha}{K\sin\overset{o}{\tau}}\ ,
\end{aligned}
\tag{B1}
$$

$$
(g_{\theta\theta},\ r^2) = -1 \quad \rightarrow \overset{o}{\tau}_{,\theta} = \pm\frac{\sqrt{K}}{q}\ \text{(we will use the sign +)}\ ,
\tag{B2}
$$

$$
(g_{ur},\ r^0) = 1 \quad \rightarrow \overset{o}{\pi}_{,u} = \frac{1}{\beta q}\ ,
\tag{B3}
$$

$$
\begin{aligned}
(g_{r\phi},\ r^0) &= 0 && \rightarrow \overset{1}{f} = \frac{a_0\alpha}{q\sin^2\overset{o}{\tau}}\,\frac{\overset{1}{\tau}\,q\sin\overset{o}{\tau}+\overset{o}{\pi}\cos\overset{o}{\tau}}{K}\ , \\
(g_{r\theta},\ r^0) &= 0 && \rightarrow \overset{1}{\tau} = -\frac{1}{\overset{o}{\tau}_{,\theta}\,q\sin\overset{o}{\tau}}\Big[\overset{o}{\pi}_{,\theta}\,\beta K\sin\overset{o}{\tau}+\overset{o}{\pi}\overset{o}{\tau}_{,\theta}\cos\overset{o}{\tau}\,(1-\beta K)\Big]\ , \\
(g_{r\phi},\ r^0) &= 0 && \rightarrow \overset{1}{f} = \frac{a_0\alpha\beta}{q\,\overset{o}{\tau}_{,\theta}\sin^2\overset{o}{\tau}}(\overset{o}{\pi}\overset{o}{\tau}_{,\theta}\cos\overset{o}{\tau}-\overset{o}{\pi}_{,\theta}\sin\overset{o}{\tau})\ ,
\end{aligned}
\tag{B4}
$$

$$(g_{u\theta},\ r^1) = 0 \quad \to \overset{1}{\tau}_{,u} = \frac{1}{\beta q^3\, \overset{o}{\tau}_{,\theta} \sin \overset{o}{\tau}}\left[q_{,\theta}\, \beta K \sin \overset{o}{\tau} + \overset{o}{\tau}_{,\theta}\, q \cos \overset{o}{\tau}\,(-1+\beta K)\right] , \tag{B5}$$

$$\begin{aligned}(g_{\theta\theta}g_{\phi\phi},\ r^3) = 0 \quad \to \overset{o}{\sigma} = {} & \frac{1}{2\overset{o}{\tau}_{,\theta}^{\ 3} \sin^2 \overset{o}{\tau}}\Big\{\overset{o}{\tau}_{,\theta}^{\ 3}(1-2\sin^2\overset{o}{\tau})\, \overset{o}{\pi}\,(1-\beta K) + K\sin\overset{o}{\tau}\,\Big[\beta\, \overset{o}{\pi}_{,\theta}\,(\overset{o}{\tau}_{,\theta}^{\ 2}\cos\overset{o}{\tau} - \overset{o}{\tau}_{,\theta\theta}\sin\overset{o}{\tau}) \\ & + \overset{o}{\tau}_{,\theta}\Big(\beta_{,\theta}\,(\overset{o}{\pi}_{,\theta}\sin\overset{o}{\tau} - \overset{o}{\pi}\overset{o}{\tau}_{,\theta}\cos\overset{o}{\tau}) + \beta\, \overset{o}{\pi}_{,\theta\theta}\sin\overset{o}{\tau}\Big)\Big]\Big\} ,\end{aligned}$$

$$(g_{\phi\phi},\ r^2) = -\sin^2\theta \quad \to \sin\overset{o}{\tau} = \pm\frac{\sin\theta}{q\sqrt{K}} , \tag{B6}$$

$$\begin{aligned}(g_{\phi\phi},\ r^1) = 2c\sin^2\theta \quad \to c = {} & -\frac{1}{2q\sin^2\overset{o}{\tau}\,(1+a_0^2\alpha^2)}\Big[2\sin\overset{o}{\tau}\,(1+a_0^2\alpha^2)(\overset{o}{\sigma}\sin\overset{o}{\tau} + \overset{1}{\tau}\, q\cos\overset{o}{\tau}) \\ & + \mu_1 q\sin^2\overset{o}{\tau}\,(-1+a_0^2\alpha^2) + 2a_0\alpha^2(a_0\,\overset{o}{\pi} + a_1 q\sin^2\overset{o}{\tau})\Big] \\ \to c_{,u} = {} & -\frac{a_0\alpha}{K}a_{1,u} + \frac{1-a_0^2\alpha^2}{2(1+a_0^2\alpha^2)}\mu_{1,u} - \frac{q_{,\theta}^{\ 2}}{2q^2} \\ & - \frac{q_{,\theta}\cot\overset{o}{\tau}\sqrt{K}}{q^2} + \frac{1-a_0^2\alpha^2}{2q^2\beta\sin^2\overset{o}{\tau}\,(1+a_0^2\alpha^2)} - \frac{1}{2} - \frac{K\cot^2\overset{o}{\tau}}{2q^2} ,\end{aligned} \tag{B7}$$

$$\begin{aligned}(g_{\theta\phi},\ r^1) = -2d\sin\theta \quad \to d = {} & -\frac{1}{2qK\sin^2\overset{o}{\tau}}\Big[(1-a_0^2\alpha^2)q\sin^2\overset{o}{\tau}\, a_1 + 2qa_0\mu_1\sin^2\overset{o}{\tau} + 2a_0\,\overset{o}{\pi}\Big] \\ \to d_{,u} = {} & -\frac{a_0}{q^2K\beta\sin^2\overset{o}{\tau}} - \frac{a_0}{K}\mu_{1,u} - \frac{1-a_0^2\alpha^2}{2K}a_{1,u} \ .\end{aligned} \tag{B8}$$

Equations $(g_{uu},\ r^0) = 1$, $(g_{u\theta},\ r^2) = 0$, $(g_{u\phi},\ r^1) = 0$, $(g_{rr},\ r^{-1}) = 0$, $(g_{r\theta},\ r^1) = 0$, $(g_{r\phi},\ r^1) = 0$, and $(g_{\theta\theta},\ r^1) = -2c$ are satisfied identically.

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[1] J. Bičák and B. G. Schmidt, *Isometries compatible with gravitational radiation*, J. Math. Phys. **25**, 600 (1984).
[2] J. Bičák and A. Pravdová, *Symmetries of asymptotically flat electrovacuum space-times and radiation*, J. Math. Phys. **39**, 6011 (1998).
[3] V. Pravda and A. Pravdová, *Boost-rotation symmetric spacetimes - review*, Czech. J. Phys. **50**, 333 (2000); see also gr-qc/0003067.
[4] J. Bičák, *Selected solutions of Einstein's field equations: their role in general relativity and astrophysics*, in *Einstein's Field Equations and Their Physical Meaning*, ed. B. G. Schmidt, Springer Verlag, Berlin – New York (2000).
[5] J. F. Plebański and M. Demiański, *Rotating, charged and uniformly accelerating mass in general relativity*, Ann. Phys. (USA) **98**, 98 (1976).
[6] J. Bičák and V. Pravda, *Spinning C-metric as a boost-rotation symmetric radiative spacetime*, Phys. Rev. D **60**, 044004 (1999).
[7] P. S. Letelier and S. R. Oliveira, *On uniformly accelerated black holes*, Phys. Rev. D, at press (2001).
[8] P. S. Letelier, *Spinning strings as torsion line spacetime defects*, Class. Quantum Grav. **12**, 471 (1995).
[9] W. B. Bonnor, *The interactions between two classical spinning particles*, Class. Quantum Grav. **18**, 1382 (2001).
[10] J. Bičák and B. G. Schmidt, *Asymptotically flat radiative space-times with boost-rotation symmetry: The general structure*, Phys. Rev. D **40**, 1827 (1989).
[11] J. Bičák, *Gravitational radiation from uniformly accelerated particles in general relativity*, Proc. Roy. Soc. A **302**, 201 (1968).
[12] J. Bičák, *Radiative properties of space-times with the axial and boost symmetries*, in *Gravitation and Geometry*, Eds. W. Rindler and A. Trautman, Bibliopolis, Naples (1987).
[13] R. Lazkoz and A. Valiente-Kroon, *Boost-rotation symmetric type D radiative metrics in Bondi coordinates*, Phys. Rev. D **62**, 084033 (2000).

[14] H. Bondi, M. G. J. van der Burg, and A. W. K. Metzner, *Gravitational waves in general relativity VII. Waves from axi-symmetric isolated systems*, Proc. Roy. Soc. A **269**, 21 (1962).
[15] R. K. Sachs, *Gravitational waves in general relativity VIII. Waves in asymptotically flat space-time*, Proc. Roy. Soc. A **270**, 103 (1962).
[16] M. G. J. van der Burg, *Gravitational waves in general relativity X. Asymptotic expansions for the Einstein-Maxwell field*, Proc. Roy. Soc. A **310**, 221 (1969).
[17] D. Kramer, H. Stephani, E. Herlt, and M. A. H. MacCallum, *Exact solutions of Einstein's field equations*, Ed. E. Schmutzer, Cambridge Univ. Press, Cambridge (1980).